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THE FOURIER EXPANSION OF EISENSTEIN SERIES FOR GL(3, Z)

BY

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ABSTRACT. The Fourier expansions of Eisenstein series for GL(3, **Z**) are obtained by two methods—one analogous to the classical method used by many number theorists, including Weber, in his derivation of the Kronecker limit formula. The other method is analogous to that used by Siegel to obtain Fourier expansions of Eisenstein series for the Siegel modular group. The expansions involve matrix argument K-Bessel functions recently studied by Tom Bengtson. These K-Bessel functions are natural generalizations of the ordinary K-Bessel function which arise when considering harmonic analysis on the symmetric space of the general linear group using a certain system of coordinates.

1. Introduction and preliminaries. Automorphic forms for $\mathrm{GL}(n, \mathbf{Z})$ are analogues of Siegel modular forms (cf. **[16]**), which are necessary for harmonic analysis on the fundamental domain of Minkowski-reduced positive matrices $P_n/\mathrm{GL}(n, \mathbf{Z})$. Here $P_n =$ the symmetric space of positive definite $n \times n$ real symmetric matrices Y. A matrix A in the general linear group of nonsingular $n \times n$ real matrices, i.e., $A \in \mathrm{GL}(n, \mathbf{R})$, acts on $Y \in P_n$ via $Y \mapsto {}^tAYA = Y[A]$, where ${}^tA =$ the transpose of A. The action is transitive so that $P_n \cong O(n) \setminus \mathrm{GL}(n, \mathbf{R})$, with $O(n) =$ the orthogonal group of all $n \times n$ matrices V with ${}^tV = V^{-1}$. Applications of harmonic analysis on $P_n/\mathrm{GL}(n, \mathbf{Z})$ include the extension of Hecke's correspondence between modular forms and Dirichlet series to Seigel's modular forms (cf. Imai **[8]**) and the generalization of Selberg's trace formula to $\mathrm{GL}(n, \mathbf{Z})$, with all the consequences that entails (cf. Selberg **[15]**).

Here we consider the Fourier expansions of Eisenstein series for $\mathrm{GL}(3, \mathbf{Z})$. These results were obtained by the first author using the Bruhat decomposition (Theorem 1). The second author got into the act by finding a different approach which does not use the Bruhat decomposition. Instead Theorem 2 is required. The two methods lead to two different Fourier expansions—Theorems 3 and 4. The two approaches to the Fourier expansion are the analogues of standard methods for nonholomorphic Eisenstein series for $\mathrm{SL}(2, \mathbf{Z})$. Thus, the Bruhat decomposition is used by Kubota in **[10]**, while Bateman and Grosswald use the analogue of Theorem 2 in **[2]**. For $\mathrm{SL}(2, \mathbf{Z})$, the equality of the two types of Fourier expansions is a simple formula

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relating a singular series involving Ramanujan sums and the divisor function (cf. Hardy [**6**, p. 141]):

$$\zeta(2s)\sum_{\substack{c>0\\ d \bmod c\\ (d,c)=1}} c^{-2s}e^{2\pi i m d/c}=\sigma_{1-2s}(m)\equiv\sum_{0<d/m} d^{1-2s}. \tag{1.1}$$

The Eisenstein series for SL(2, **Z**) is *Epstein's zeta function*:

$$Z(s\mid Y)=\frac{1}{2}\sum_{a\in\mathbf{Z}^2-0} Y[a]^{-s},\qquad \operatorname{Re} s>1,\ Y\in P_2. \tag{1.2}$$

The *Fourier expansion of Epstein's zeta function* has been of great use in number theory. It says that

$$\begin{aligned}\pi^{-s}\Gamma(s)Z(s\mid Y)&=y^s\Lambda(s)+y^{1-s}\Lambda(1-s)\\ &\quad+2\sum_{n\neq 0}e^{2\pi i n x}|n|^{s-1/2}\sigma_{1-2s}(n)y^{1/2}K_{s-1/2}(2\pi|n|y),\end{aligned} \tag{1.3}$$

where

$$\Lambda(s)=\pi^{-s}\Gamma(s)\zeta(2s)\quad\text{and}\quad K_s(y)=\text{the } K\text{-Bessel function}. \tag{1.4}$$

Weber used this expansion to derive Kronecker's limit formula in [**25**, III, pp. 526 ff]. Stark used a generalization of it to show that there are exactly 9 imaginary quadratic fields of class number one in [**17**]. See also Chowla and Selberg [**4**], Weil [**26**] and Terras [**19**].

There are many analogues of (1.3) for discrete groups Γ acting on symmetric spaces G/K. For example, Siegel considers $\Gamma=\mathrm{Sp}(n,\mathbf{Z})$ acting on $\mathrm{Sp}(n,\mathbf{R})/U(n)$ in [**16**]. Since the Fourier coefficients of Siegel's (holomorphic) Eisenstein series are rational numbers with bounded denominators, one can show that the Satake compactification of $U(n)\backslash\mathrm{Sp}(n,\mathbf{R})/\mathrm{Sp}(n,\mathbf{Z})$ is defined over $\mathbf{Q}$ (cf. Baily [**1**, p. 238]). Fourier expansions of nonholomorphic Eisenstein series for $\mathrm{Sp}(n,\mathbf{Z})$ have been obtained by Maass in [**14**, Chapter 18]. The arithmetic part of the Fourier coefficients of holomorphic and nonholomorphic Eisenstein series for $\mathrm{Sp}(n,\mathbf{Z})$ is called a singular series. The nonarithmetic part is $\exp\{-\mathrm{Tr}(NY)\}$, for $Y\in P_n$, ${}^tN=N=$ half-integer matrix, in the holomorphic case. But in the nonholomorphic case the nonarithmetic part is a matrix integral analogue of a confluent hypergeometric function. Eisenstein series for congruence subgroups of $\mathrm{SL}(2,\mathcal{O}_k)$, $\mathcal{O}_k=$ the ring of integers of a totally imaginary number field, have Fourier expansions which have been used by Kubota [**11**], and Heath-Brown and Patterson [**7**] to study cubic Gauss sums.

Thus it is of interest to compute the Fourier coefficients of Eisenstein series for GL(3, **Z**). Here we give a discussion analogous to the classical developments outlined above. This is possible because there is a well developed theory of Bessel functions for P_n, due to Bengtson in [**3**]. Langlands gives an elegant theory of the constant terms of these Fourier expansions in [**12**], but does not discuss the other terms. Jacquet, Piatetski-Shapiro and Shalika have taken an abstract adelic approach to the theory of automorphic forms for GL(3) using Whittaker models in [**9**].

Matrix K-Bessel functions for P_n are functions $f\colon P_n \to \mathbf{C}$ which are eigenfunctions for the $\mathrm{GL}(n, \mathbf{R})$-invariant differential operators on P_n, satisfying the invariance property

$$(1.5)\qquad f\left(Y\begin{bmatrix} I_m & X \\ 0 & I_{n-m}\end{bmatrix}\right) = e^{2i\operatorname{Tr}({}^tNX)}f(Y),$$

where $I_m = m \times m$ identity matrix $X \in \mathbf{R}^{m\times(n-m)}$, $Y \in P_n$, assuming also that $f(Y)$ has at most polynomial growth in $|Y_j|$, $Y = \left(\begin{smallmatrix} Y_j & * \\ * & *\end{smallmatrix}\right)$, $Y_j \in P_j$, $|Y_j| =$ determinant Y_j.

The classical K-Bessel function can easily be viewed this way (cf. Terras **[22]**).

In order to give an integral formula for matrix K-Bessel functions, we need to define the *power function* for $s = (s_1, \dots, s_m) \in \mathbf{C}^m$, $Y \in P_n$, $m \leqslant n$;

$$(1.6)\qquad p_s(Y) = \prod_{j=1}^{m} |Y_j|^{s_j}, \quad Y = \begin{pmatrix} Y_j & * \\ * & * \end{pmatrix}, \qquad Y_j \in P_j,$$

$|Y_j| =$ determinant Y_j. This is the basic eigenfunction for all the $\mathrm{GL}(n, \mathbf{R})$-invariant differential operators on P_n if $m = n$ (cf. Maass **[14]** or Terras **[18]**). Harish-Chandra has shown that the $O(n)$-invariant eigenfunctions of the $\mathrm{GL}(n, \mathbf{R})$-invariant differential operators on P_n are formed by averaging power functions over $O(n)$ (cf. Terras **[18]**). Analogous ideas work to build up functions with the invariance property (1.5).

The *first definition of a matrix argument K-Bessel function* of $s \in \mathbf{C}^m$, with $\operatorname{Re} s_i$ in a suitable half-plane, $Y \in P_n$, $N \in \mathbf{R}^{m\times(n-m)}$, $0 < m < n$, is

$$(1.7)\qquad k_{m,n-m}(s \mid Y, N) = \int_{X\in\mathbf{R}^{m\times(n-m)}} p_{-s}\left(Y^{-1}\begin{bmatrix} I_m & 0 \\ {}^tX & I_{n-m}\end{bmatrix}\right) e^{2i\operatorname{Tr}({}^tNX)}\, dX.$$

Note that we do not allow the power function $p_s(Y)$ to have all of its possible n complex variables. It can easily be seen that if $y > 0$, then

$$(1.8)\qquad \Gamma(s)k_{1,1}\left(s \,\middle|\, \begin{pmatrix} 1/y & 0 \\ 0 & y\end{pmatrix}, \pi n\right) = \begin{cases} 2\pi^s |n|^{s-1/2} y K_{s-1/2}(2\pi |n| y), & n \neq 0, \\ y^{1-s}\Gamma(1/2)\Gamma(s - 1/2), & n = 0. \end{cases}$$

Here $K_s(y)$ is the usual K-Bessel function, as in (1.3). Note, for future reference, that

$$k_{1,1}\left(s \,\middle|\, \begin{pmatrix} 1/y & 0 \\ 0 & y\end{pmatrix}, \pi n\right) = y^{1-s}k_{1,1}(s \mid I_2, \pi n y), \quad \text{if } y > 0.$$

It is absurdly easy to read off the differential equations and invariance property (1.5) of $k_{m,n-m}$ from (1.7).

We introduce another type of matrix K-Bessel function to study convergence and Mellin transforms. When A and B are symmetric $m \times m$ real matrices such that A or B lies in P_m and $s \in \mathbf{C}^m$ (suitably restricted, if necessary), we define the *second matrix argument K-Bessel function* by

$$K_m(s \mid A, B) = \int_{W\in P_m} p_s(X)\exp\{-\operatorname{Tr}(AX + BX^{-1})\}\frac{dX}{|X|^{(m+1)/2}}.$$

It is easy to see that this integral converges for all $s \in \mathbf{C}^m$ if A and B are both in P_m. The special case $B = 0$ is

$$(1.10)\qquad K_m(s \mid A, 0) = p_s(A^{-1})\pi^{m(m-1)/4} \prod_{j=1}^{m} \Gamma\left(s_j + \cdots + s_m - \frac{j-1}{2}\right).$$

The two K-Bessel functions are related by *Bengtson's first formula* (cf. [3]):

$$(1.11)\qquad \Gamma_m(-s^*)k_{m,n-m}\left(s \mid \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix}, N\right) = \pi^{m(n-m)/2} |H|^{m/2} K_m(\tilde{s} \mid H[{}^tN], G),$$

where $s^* = (s_{m-1}, \ldots, s_1, -(s_1 + \cdots + s_m))$, $\tilde{s} = -s + (0, \ldots, 0, (n-m)/2)$. We shall need two more of Bengtson's formulas. The proofs of all these results are in [3]. *Bengtson's second formula* is

$$(1.12)\qquad k_{m,1}\left(s \mid \begin{pmatrix} G & 0 \\ 0 & h \end{pmatrix}, n\right) = p_{-s}(G^{-1}) |G|^{-1/2} h^{m/2} k_{m,1}(s \mid I_{m+1}, h^{1/2}Tn),$$

if $G^{-1} = {}^tTT$, where T is upper triangular and positive on the diagonal. *Bengston's third formula* is

$$(1.13)\qquad k_{m,1}((s_1, s_2) \mid I_{m+1}, (a, b)) = \int_{u \in \mathbf{R}} (1 + u^2)^{(m-1)/2 - \sum_{j=1}^{m} s_j} k_{m-1,1}\left(s_2 \mid I_m, b\sqrt{1+u^2}\right) e^{2iau}\, du,$$

for $s_1 \in \mathbf{C}$, $s_2 \in \mathbf{C}^{m-1}$, $a \in \mathbf{R}$, $b \in \mathbf{R}^{m-1}$, and suitably restricted. Formula (1.13) was first proved by K. Imai in the special case $m = 2$, which is all that is needed here.

The prerequisite for deducing Fourier expansions of Eisenstein series by the method of Siegel [**16**], Maass [**14**, pp. 300–308], Baily [**1**, pp. 228–240] and Terras [**21**] is the Bruhat decomposition of $\mathrm{SL}(n, \mathbf{Q})$ with respect to *the maximal parabolic subgroup*

$$(1.14)\qquad P_{\mathbf{Q}} = P(n-1, 1)_{\mathbf{Q}} = \left\{ \begin{pmatrix} A & B \\ 0 & c \end{pmatrix} \;\middle|\; A \in \mathrm{GL}(n-1, \mathbf{Q}), c \in \mathbf{Q} - 0, B \in \mathbf{Q}^{n-1} \right\}.$$

The *Bruhat decomposition* with respect to P is

$$(1.15)\qquad \mathrm{SL}(n, \mathbf{Q}) = P_{\mathbf{Q}} \cup P_{\mathbf{Q}}\sigma P_{\mathbf{Q}} \text{ (disjoint union)}, \quad \text{where } \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-2} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

To prove (1.15), multiply matrices and note that a matrix $\left(\begin{smallmatrix} E & F \\ g & h \end{smallmatrix}\right) \in \mathrm{SL}(n, \mathbf{Q})$ lies in $P_{\mathbf{Q}}\sigma P_{\mathbf{Q}}$ if and only if rank $g = 1$, assuming $E \in \mathbf{Q}^{(n-1)\times(n-1)}$, $h \in \mathbf{Q}$. The general theory of Bruhat decompositions is described very well in Curtis [**5**].

Next we must use the Bruhat decomposition to obtain coset representatives for the cosets to be summed over in the Eisenstein series for $\mathrm{GL}(3, \mathbf{Z})$.

THEOREM 1 (A SET OF COSET REPRESENTATIVES À LA BRUHAT). *Suppose P is defined by* (1.14). *The cosets in* $\mathrm{SL}(n,\mathbf{Z})/P_{\mathbf{Z}}$ *can be represented by* $S_1^* \cup S_2^*$, *where*

$$S_1^* = \{I\},$$

$$S_2^* = \left\{ \begin{pmatrix} {}^tA^{-1} & 0 \\ 0 & 1 \end{pmatrix} n_q \sigma p_q \;\middle|\; \begin{array}{c} A \in \mathrm{SL}(n-1,\mathbf{Z})/P(1,n-2); \\ q = e/f,\ f \geq 1, (e,f) = 1; \\ e, f \in \mathbf{Z}; \end{array} \right\},$$

with

$$n_q = \begin{pmatrix} 1 & 0 & q \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad p_q = \begin{pmatrix} f & 0 & g \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1/f \end{pmatrix},$$

if $eg \equiv 1 \pmod f$, $0 \leq g < f$.

PROOF. The general idea is to follow the proof in Baily **[1]** and Terras **[21]** with a more explicit version of p_u. Define

$$T\colon P_{\mathbf{Q}} \to (\mathrm{SL}(n,\mathbf{Z}) \cap P_{\mathbf{Q}}\sigma P_{\mathbf{Q}}/P_{\mathbf{Z}},$$

by $T(p) = p\sigma p' \pmod{P_{\mathbf{Z}}}$, for $p \in P_{\mathbf{Q}}$. Here p' is chosen in $P_{\mathbf{Q}}$ to put $p\sigma p' \in \mathrm{SL}(n,\mathbf{Z})$. This is possible by Lemma 2.2 of Terras **[21]**.

Then matrix multiplication shows $\sigma p = p'\sigma$ is equivalent to

$$p = \begin{pmatrix} a & 0 & 0 \\ c & D & e \\ 0 & 0 & g \end{pmatrix}, \quad \text{with } D \in \mathbf{Q}^{(n-2)\times(n-2)}.$$

So define

$$P_{\mathbf{Q}}^* = \left\{ p = \begin{pmatrix} a & 0 & 0 \\ c & D & 0 \\ 0 & 0 & g \end{pmatrix} \;\middle|\; p \in \mathrm{SL}(n,\mathbf{Q}),\ D \in \mathbf{Q}^{(n-2)\times(n-2)} \right\}.$$

Finding the coset representatives for $\mathrm{SL}(n,\mathbf{Z})/P_{\mathbf{Z}}$ is the same as reducing $p \in P_{\mathbf{Q}}$ modulo $P_{\mathbf{Q}}^*$. Representatives for $P_{\mathbf{Q}}/P_{\mathbf{Q}}^*$ are

$$\begin{pmatrix} {}^tA^{-1} & {}^tA^{-1}C \\ 0 & 1 \end{pmatrix}, \qquad A \in \mathrm{SL}(n-1,\mathbf{Z})/P(1,n-1)_{\mathbf{Z}},\ C = \begin{pmatrix} q \\ 0 \end{pmatrix}, q \in \mathbf{Q}.$$

The equality $p\sigma p' = p_1\sigma p_1'$ with $p, p', p_1, p_1' \in P_{\mathbf{Q}}$ implies $p_1^{-1}p \in P_{\mathbf{Q}}^*$. Thus, if

$$p = \begin{pmatrix} {}^tA^{-1} & 0 \\ 0 & 1 \end{pmatrix} n_q,$$

then $T(p) = p\sigma p'$ gives a complete set of representatives for

$$(\mathrm{SL}(n,\mathbf{Z}) \cap P_{\mathbf{Q}}\sigma P_{\mathbf{Q}})/P_{\mathbf{Z}}.$$

Finally, it must be proved that if $q = e/f$, $f \geq 1$, $(e,f) = 1$, then

$$p' = \begin{pmatrix} f & 0 & g \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1/f \end{pmatrix}, \quad \text{with } eg \equiv 1 \pmod f.$$

To see this, write

$$ {}^tA^{-1} = \begin{pmatrix} a & b \\ c & D \end{pmatrix}, \qquad D \in \mathbf{Z}^{(n-2)\times(n-2)}. $$

Then

$$ \begin{pmatrix} {}^tA^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & q \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-2} & 0 \\ -1 & 0 & 0 \end{pmatrix} p' = \begin{pmatrix} -ae & b & a(-qg+1/f) \\ -ce & d & c(-qg+1/f) \\ -f & 0 & -g \end{pmatrix}. $$

Clearly the matrix on the right lies in $\mathrm{SL}(n,\mathbf{Z})$, assuming that $q = e/f$ is as stated in the theorem. Q.E.D.

In order to obtain Fourier expansions of Eisenstein series by a method analogous to that of Bateman and Grosswald [2] and Terras [20] for Epstein's zeta function, we need

THEOREM 2 (COSET REPRESENTATIVES SANS BRUHAT). *The cosets of*

$$ \mathbf{Z}^{n\times(n-1)}\mathrm{rank}(n-1)/\mathrm{GL}(n-1,\mathbf{Z}) $$

can be represented by $S_1 \cup S_2$, *where*

$$ S_1 = \left\{ \begin{pmatrix} B \\ 0 \end{pmatrix} \,\middle|\, B \in \mathbf{Z}^{(n-1)\times(n-1)}\mathrm{rank}(n-1)/\mathrm{GL}(n-1,\mathbf{Z}) \right\}, $$

and

$$ S_2 = \left\{ \begin{pmatrix} B \\ {}^tg \end{pmatrix} \,\middle|\, \begin{array}{l} B = {}^tA^{-1}D, \quad A \in \mathrm{SL}(n-1,\mathbf{Z})/P, \quad {}^tg = (c,0,\ldots,0); \\ D = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ d_{ij} & & \ddots & \\ & & & d_{n-1} \end{pmatrix}, \quad \begin{array}{l} c > 0, \\ d_2 > 0,\ldots,d_{n-1} > 0, \\ d_{ij} \bmod d_j,\ j = 2,\ldots,n-1 \end{array} \end{array} \right\}, $$

with

$$ P = \left\{ \begin{pmatrix} p_1 & & & 0 \\ & p_2 & & \\ * & & \ddots & \\ & & & p_{n-1} \end{pmatrix} \in \mathrm{GL}(n-1,\mathbf{Z}) \right\}. $$

PROOF. See the proof of formula (2.3.11) in Terras [23].

2. Fourier expansions. It will be convenient to consider automorphic forms on the determinant one surface $\mathrm{SP}_n = \{Y \in P_n \mid |Y| = 1\}$, which is the symmetric space of the special linear group $\mathrm{SL}(n,\mathbf{R}) = \{A \in \mathrm{GL}(n,\mathbf{R}) \mid |A| = 1\}$. Let $D(\mathrm{SP}_n)$ denote the $\mathrm{SL}(n,\mathbf{R})$-invariant differential operators on SP_n. Suppose Γ is some subgroup of

$\mathrm{GL}(n,\mathbf{Z})$. We define an *automorphic form* φ for Γ to be a function $\varphi\colon \mathrm{SP}_n \to \mathbf{C}$ such that

$$(2.1)\qquad \begin{cases} (1)\quad \varphi(Y[A]) = \varphi(Y), \quad \text{for all } Y \in \mathrm{SP}_n, A \in \Gamma; \\ (2)\quad L\varphi = \chi(L)\varphi, \quad \text{for all } L \in D(\mathrm{SP}_n), \text{ with } \chi(L) \in \mathbf{C}; \\ (3)\quad \varphi(Y) \text{ has at most polynomial growth in } |Y_j|; \\ \qquad\qquad \text{where } Y = \begin{pmatrix} Y_j & * \\ * & * \end{pmatrix}, Y_j \in P_j, \text{ as } |Y_j| \to \infty;\ j = 1,\ldots,n-1. \end{cases}$$

We shall write $\varphi \in \mathcal{Q}(\Gamma, \chi)$ if (2.1) holds.

These automorphic forms were first introduced by Maass in **[13]** for the case $n = 2$. The general definition was given by Harish-Chandra (for a much more general situation than we shall consider here). There are still many mysteries about these automorphic forms, but we want to emphasize that they are also quite analogous to holomorphic Siegel modular forms (cf. Siegel **[16]**). In particular, we shall obtain Fourier expansions of Eisenstein series in $\mathcal{Q}(\mathrm{GL}(3,\mathbf{Z}), \lambda)$ by methods which are natural extensions of those of Siegel in **[16]**. Suppose that $\varphi \in \mathcal{Q}(\mathrm{SL}(2,\mathbf{Z}), \chi)$, $Y \in \mathrm{SP}_3$, $s \in \mathbf{C}$ with $\mathrm{Re}\, s > 3/2$, $W^0 = |W|^{-1/2}W$ for $W \in P_2$. Then we define *the Eisenstein series* associated to s, and the parabolic subgroup $P = P(2,1)$ of $\mathrm{SL}(3,\mathbf{Z})$ (as in (1.14)) by

$$(2.2)\qquad E(s,\varphi \mid Y) = \sum_{A = (A_1 *) \in \mathrm{SL}(3,\mathbf{Z})/P(2,1)} |Y[A_1]|^{-s} \varphi\big(Y[A_1]^0\big).$$

When φ is an eigenfunction of all the Hecke operators for $\mathrm{SL}(2,\mathbf{Z})$ (cf. **[24]**), then there is a relation between the Eisenstein series and the following *zeta function*:

$$(2.3)\qquad Z(s,\varphi \mid Y) = \sum_{B \in \mathbf{Z}^{3\times 2} \mathrm{rk}\, 2/\mathrm{GL}(2,\mathbf{Z})} |Y[B]|^{-s} \varphi\big(Y[B]^0\big),$$

with s, φ, Y as in (2.2). *The relation between the Eisenstein series and the zeta function is*

$$(2.4)\qquad Z(s,\varphi \mid Y) = L\varphi(2s) E(s,\varphi \mid Y),$$

assuming that φ is an eigenfunction of all the Hecke operators T_m for $\mathrm{GL}(2,\mathbf{Z})$ as defined by formula (2.1) of Terras **[24]**. If $T_m\varphi = u_m\varphi$ for all $m \geq 1$, then the *L-function* associated to $\varphi \in \mathcal{Q}(\mathrm{SL}(2,\mathbf{Z}), \chi)$ is

$$(2.5)\qquad L\varphi(s) = \sum_{m \geq 1} u_m m^{-s}, \quad \text{for } \mathrm{Re}\, s > 1.$$

This L-function has an Euler product and analytic continuation with functional equation (cf. Maass **[14]** and Terras **[24]**). The proof of (2.4) is in Terras **[24]**.

In order to obtain the Fourier expansions of Eisenstein series, we must use that of $\varphi \in \mathcal{Q}(\mathrm{SL}(2,\mathbf{Z}), \chi)$. We shall write this in a slightly unusual way to simplify the computation. First write the Iwasawa decomposition of $U \in \mathrm{SP}_2$:

$$(2.6)\qquad U = \begin{pmatrix} y & 0 \\ 0 & 1/y \end{pmatrix} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}, \qquad y > 0, x \in \mathbf{R}.$$

Note that this allows us to identify SP_2 with $H = \{x + iy \in \mathbf{C} \mid y > 0\}$:

$$H \to SL(2,\mathbf{R})/SO(2) \to SP_2,$$

$$x + iy \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} i = gi$$

$$\mapsto g^t g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1/y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = U.$$

Since $\varphi(U)$ is a periodic function of x in (2.6), property (1) of (2.1) implies that φ has the *Fourier expansion*

$$\varphi(U) = \alpha_0 y^r k_{1,1}(1 - r \mid I_2, 0) + \alpha_0' y^{1-r} k_{1,1}(r \mid I_2, 0) + \sum_{n \neq 0} \exp\{2\pi i n x\} \alpha_n y^{1-r} k_{1,1}(r \mid I_2, \pi n y), \tag{2.7}$$

where r is determined from the differential equation

$$y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \varphi = r(r-1)\varphi. \tag{2.8}$$

Note that

$$\varphi(U) = \alpha_0 k_{1,1}(1 - r \mid U, 0) + \alpha_0' k_{1,1}(r \mid U, 0) + \sum_{n \neq 0} \alpha_n k_{1,1}(r \mid U, \pi n).$$

These Fourier expansions are a simple consequence of the uniqueness property of $k_{1,1}$. This can also be viewed as separation of variables in the PDE (2.8) (cf. Terras [**18**]). When φ is the Eisenstein series

$$\varphi = \varphi_r(U) = \pi^{-r}\Gamma(r) Z(r \mid U), \tag{2.9}$$

with $Z(r \mid U) =$ Epstein's zeta function (1.2), we obtain

$$\alpha_0 = \Lambda(r)/B(\tfrac{1}{2}, \tfrac{1}{2} - r), \qquad \alpha_0' = \Lambda(1 - r)/B(\tfrac{1}{2}, r - \tfrac{1}{2}), \\ \alpha_n = \pi^{-r}\Gamma(r)\sigma_{1-2r}(n), \qquad n \neq 0, \tag{2.10}$$

with $\sigma_r(n)$ and $\Lambda(r)$ as in (1.1) and (1.4). Here $B(a, b)$ is the beta function $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$. Thus formula (2.10) is just a restatement of (1.3) using the fact that

$$k_{1,1}(r \mid I_2, 0) = B(\tfrac{1}{2}, r - \tfrac{1}{2}). \tag{2.11}$$

We first develop the Fourier expansion of $E(s, \varphi \mid Y)$ using the Bruhat-type coset representatives of Theorem 1, assuming that $\varphi \equiv \varphi_r \in \mathcal{Q}(SL(2, \mathbf{Z}), r(r - 1))$ is a *cusp form*; i.e., that $\alpha_0 = \alpha_0' = 0$ in the Fourier expansion (2.7).

THEOREM 3. *Suppose that* $\varphi_r \in \mathcal{Q}(SL(2, \mathbf{Z}), r(r - 1))$ *is a cusp form so that* $\alpha_0 = \alpha_0' = 0$ *in its Fourier expansion* (2.7) *and the other Fourier coefficients are* α_k, $k \neq 0$. *Then the Fourier expansion of* $E(s, \varphi_r \mid Y)$ *for*

$$Y = \begin{pmatrix} U & 0 \\ 0 & w \end{pmatrix} \begin{bmatrix} I_2 & x \\ 0 & 1 \end{bmatrix}, \qquad U \in P_2, w = |U|^{-1}, x \in \mathbf{R}^2,$$

as a periodic function of $x \in \mathbf{R}^2$ *is*

$$E(s, \varphi_r \mid Y) = |U|^{-s}\varphi_r(U^0) + \sum_{k,n,A,f} \exp\{2\pi i\, {}^t x A m\} c_f(n)\alpha_k f^{-2s+1-r}$$
$$\times k_{2,1}\left(s - \frac{r}{2}, r \,\middle|\, \begin{pmatrix} U[{}^tA^{-1}] & 0 \\ 0 & w \end{pmatrix}, \pi m\right),$$

with ${}^t m = (n, -kf)$ *and the sum running over* $k \in \mathbf{Z} - 0$, $n \in \mathbf{Z}$, $f \geqslant 1$, $A \in \mathrm{SL}(2, \mathbf{Z})/P(1,1)$, *for* $P(1,1)$ *the paraboic subgroup as in* (1.14). *Here* $c_f(n)$ *is Ramanujan's sum defined by*

$$c_f(n) = \sum_{\substack{0<e<f \\ (e,f)=1}} \exp\{2\pi i n e/f\}.$$

PROOF. Set

$$\tilde{Y} = Y\begin{bmatrix} {}^tA^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} \tilde{U} & 0 \\ 0 & \tilde{w} \end{pmatrix}\begin{bmatrix} I_2 & x \\ 0 & 1 \end{bmatrix}. \tag{2.12}$$

It is easily seen that

$$\tilde{U} = U[{}^tA^{-1}], \quad \tilde{U}^{-1} = U^{-1}[A], \qquad \tilde{w} = w, \tag{2.13}$$
$$\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = {}^tAx = \begin{pmatrix} {}^ta_1 x \\ {}^ta_2 x \end{pmatrix}, \quad \text{if } A = (a_1 a_2).$$

Using Theorem 1 (with $n = 3$), we are led to compute $Y[(n_q \sigma p_q)_1]$, where the subscript "1" means that we must take the first two columns of the 3×3 matrix $n_q \sigma p_q$. Recall that $q = e/f$ and

$$(n_q \sigma p_q)_1 = \begin{pmatrix} -qf & 0 \\ 0 & 1 \\ -f & 0 \end{pmatrix}.$$

So we set

$$Y^{\#} = \tilde{Y}\left[(n_q \sigma p_q)_1\right] = \tilde{U}\begin{bmatrix} -f(q + \tilde{x}_1) & 0 \\ -f\tilde{x}_2 & 1 \end{bmatrix} + \begin{pmatrix} wf^2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.14}$$

In order to use the Fourier expansion (2.7), we must set

$$Y^{\#} = |Y^{\#}|^{1/2}\begin{pmatrix} y & 0 \\ 0 & y^{1/2} \end{pmatrix}\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} = \begin{pmatrix} * & * \\ x|Y^{\#}|^{1/2}/y & |Y^{\#}|^{1/2}/y \end{pmatrix}, \tag{2.15}$$

and

$$\tilde{U} = \begin{pmatrix} t & 0 \\ 0 & v \end{pmatrix}\begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix} = \begin{pmatrix} * & * \\ vp & v \end{pmatrix}. \tag{2.16}$$

It follows that

$$Y^{\#} = \begin{pmatrix} t & 0 \\ 0 & v \end{pmatrix}\begin{bmatrix} -f(q + \tilde{x}_1) & 0 \\ -f\{p(q + \tilde{x}_1) + \tilde{x}_2\} & 1 \end{bmatrix} + \begin{pmatrix} f^2 w & 0 \\ 0 & 0 \end{pmatrix}.$$

Putting all this together, we find that

$$(2.17)\qquad |Y^{\#}| = vf^2\{t(q+\tilde{x}_1)^2 + w\}, \qquad y = \sqrt{\frac{f^2}{v}}\sqrt{t(q+\tilde{x}_1)^2 + w},$$

$$x = -f\{p(q+\tilde{x}_1) + \tilde{x}_2\}.$$

By Theorem 1, q runs over all of $\mathbf{Q}$. So we break this sum up into a sum over $q \in \mathbf{Q}/\mathbf{Z}$ and a sum over $n \in \mathbf{Z}$. Then use Poisson summation on the variable n to see that

$$(2.18)\qquad E(s, \varphi_r \mid Y) = |U|^{-s}\varphi_r(U^0) + \sum_{A,k,q,n} \alpha_k T(s, r \mid A, k, q, n),$$

where the sum is over $A \in \mathrm{SL}(2,\mathbf{Z})/P(1,1)$, $q \in \mathbf{Q}/\mathbf{Z}$, $q = e/f$, $f \geqslant 1$, $(e, f) = 1$, $n \in \mathbf{Z}$ and

$$T = T(s, r \mid A, k, q, n) = \int_{z \in \mathbf{R}} \left(vf^2\{t(z+q+\tilde{x}_1)^2 + w\}\right)^{-s}$$

$$\times \left(\frac{f^2}{v}\{t(z+q+\tilde{x}_1)^2 + w\}\right)^{(1-r)/2}$$

$$\times k_{1,1}\left(r \mid I_2, \pi k \frac{f}{\sqrt{v}}\sqrt{t(z+q+\tilde{x}_1)^2 + w}\right)$$

$$\times \exp\left(-2\pi i k f\{p(z+q+\tilde{x}_1) + \tilde{x}_2\} - 2\pi i n z\right) dz.$$

Next let $u = \sqrt{t/w}\,(z+q+\tilde{x}_1)$ and use formula (1.13) to obtain

$$T = \exp\{2\pi i(nq + n\tilde{x}_1 - kf\tilde{x}_2)\} f^{-2s+1-r} v^{-s-(1-r)/2} t^{-1/2} w^{-s+(2-r)/2}$$

$$\times k_{2,1}\left(s - \frac{r}{2}, r \mid I_3, \left(\pi\sqrt{\frac{w}{t}}\,(kpf + n), -\pi kf\sqrt{\frac{w}{v}}\right)\right).$$

Now the last argument of $k_{2,1}$ is the vector

$$\pi\sqrt{w}\begin{pmatrix} t^{-1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix}\begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} n \\ -kf \end{pmatrix} = \pi\sqrt{w}\,M\begin{pmatrix} n \\ -kf \end{pmatrix},$$

with

$$M = \begin{pmatrix} t^{-1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix}\begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix}.$$

Now ${}^tMM = \tilde{U}^{-1} = U^{-1}[A]$. Formula (1.12) says that if $m = \binom{n}{-kf}$,

$$k_{2,1}\left(s - \frac{r}{2}, r \mid I_3, \pi\sqrt{w}\,Mm\right) = p_{s-r/2,\,r-3/2}(U^{-1}[A])\,k_{2,1}\left(s \mid \begin{pmatrix} U[{}^tA^{-1}] & 0 \\ 0 & w \end{pmatrix}, \pi m\right).$$

Next note that

$$v^{-s-(1-r)/2} w^{-s+(2-r)/2} t^{-1/2} = p_{s-r/2,\,r-3/2}(U^{-1}[A])^{-1}.$$

Thus the power functions cancel and we find that

$$T = \exp\{2\pi i(nq + n\tilde{x}_1 - kf\tilde{x}_2)\} f^{-2s+1-r}$$
$$\times k_{2,1}\left(s - \frac{r}{2}, r \,\middle|\, \begin{pmatrix} U[{}^tA^{-1}] & 0 \\ 0 & w \end{pmatrix}, \pi m\right). \quad \text{Q.E.D.}$$

Next let us use Theorem 2 to obtain an alternate Fourier expansion.

THEOREM 4. *Let $\varphi_r \in \mathcal{Q}(\mathrm{SL}(2,\mathbf{Z}), r(r-1))$ be a cusp form having Fourier expansion* (2.7) *with Fourier coefficients α_k, $k \neq 0$, and $\alpha_0 = \alpha_0' = 0$. Suppose that*

$$Y = \begin{pmatrix} U & 0 \\ 0 & w \end{pmatrix}\begin{bmatrix} I_2 & x \\ 0 & 1 \end{bmatrix}.$$

Then the Eisenstein series $Z(s, \varphi_r \mid Y)$ has the Fourier expansion:

$$Z(s, \varphi_r \mid Y) = L\varphi_r(2s)\varphi_r(U^0)\,|\,U\,|^{-s}$$
$$+ \sum_{A, C, d_1, d_2, k} \alpha_k c^{2-2s-r} d_2^{r-2s} \exp\{2\pi i\, {}^t x A m\}$$
$$\times k_{2,1}\left(s - \frac{r}{2}, r \,\middle|\, \begin{pmatrix} U[{}^tA^{-1}] & 0 \\ 0 & w \end{pmatrix}, \pi m\right),$$

where ${}^t m = c(d_1, k/d_2)$ and the sum is over $A \in \mathrm{SL}(2,\mathbf{Z})/P(1,1)$, $c > 0$, $d_1 \in \mathbf{Z}$, $0 < d_2 \mid k$, $k \neq 0$. The parabolic subgroup $P(1,1)$ is defined in (1.14).

PROOF. Everything works as in Theorem 3, except that we use the decomposition of Theorem 2 instead of Theorem 1. Define $\tilde{Y}$ as in (2.12) and (2.13). Using Theorem 2, we must set

$$\text{(2.19)} \quad Y^{\#} = \tilde{Y}\begin{bmatrix} D \\ {}^t g \end{bmatrix}, \quad \text{where } {}^t g = (c\ 0),\ c > 0,\ D = \begin{pmatrix} d_1 & 0 \\ d_{12} & d_2 \end{pmatrix},$$
$$d_1 \in \mathbf{Z},\ d_2 > 0,\ d_{12} \bmod d_2.$$

Suppose that $\tilde{U}$ is again given by (2.16). Then

$$Y^{\#} = \begin{pmatrix} t & 0 \\ 0 & v \end{pmatrix}\begin{bmatrix} d_1 + \tilde{x}_1 c & 0 \\ p(d_1 + \tilde{x}_1 c) + d_{12} + \tilde{x}_2 c & d_2 \end{bmatrix} + \begin{pmatrix} wc^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

We compute $|\,Y^{\#}\,|$, x, y in (2.15) to be

$$\text{(2.20)} \quad \begin{aligned} |\,Y^{\#}\,| &= (vd_2^2)\{t(d_1 + \tilde{x}_1 c)^2 + wc^2\}, \\ y &= \frac{1}{\sqrt{v}\, d_2}\sqrt{t(d_1 + \tilde{x}_1 c)^2 + wc^2}, \\ x &= \frac{1}{d_2}\{p(d_1 + \tilde{x}_1 c) + d_{12} + \tilde{x}_2 c\}. \end{aligned}$$

Since Theorem 2 says that the sum defining $Z(s, \varphi_r \mid Y)$ runs over all $d_1 \in \mathbf{Z}$, we can use Poisson summation to find that

$$Z(s, \varphi_r \mid Y) = L\varphi_r(2s)\varphi_r(U^0)\,|\,U\,|^{-s} + \sum \alpha_k T(s, r \mid A, c, D, k),$$

where the sum is over

$$D = \begin{pmatrix} d_1 & 0 \\ d_{12} & d_2 \end{pmatrix},$$

$d_1 \in \mathbf{Z}$, $d_2 > 0$, $d_{12} \bmod d_2$, $A \in \mathrm{SL}(2, \mathbf{Z})/P(1,1)$ and $k \neq 0$. And we define

$$\begin{aligned} T &= T(s, r \mid A, c, D, k) \\ &= \int_{z\in\mathbf{R}} \exp\left\{2\pi i\left(\frac{k}{d_2}\left[p(z + \tilde{x}_1 c) + d_{12} + \tilde{x}_2 c\right] - z d_1\right)\right\} \\ &\quad \times \left[v d_2^2\{t(z + \tilde{x}_1 c)^2 + wc^2\}\right]^{-s} \\ &\quad \times \left(\frac{t(z + x_1 c)^2 + wc^2}{v d_2^2}\right)^{(1-r)/2} k_{1,1}\left(r \mid I_2, \frac{\pi k}{\sqrt{v}\, d_2}\sqrt{t(z + x_1 c)^2 + wc^2}\right) dz. \end{aligned}$$

Now use the fact that

$$\sum_{0 < d_{12} < d_2} \exp\{2\pi i k d_{12}/d_2\} = \begin{Bmatrix} 0, & \text{if } d_2 \nmid k \\ d_2, & \text{if } d_2 \mid k \end{Bmatrix} \equiv \chi(d_2, k). \tag{2.21}$$

Therefore

$$\begin{aligned} T = {} & \chi(d_2, k)\left(v d_2^2\right)^{-s-(1-r)/2} \exp\{2\pi i(p\tilde{x}_1 + \tilde{x}_2)kc/d_2\} \\ & \times \int_{z\in\mathbf{R}} \exp\{-2\pi i z(d_1 - kp/d_2)\}\left[t(z + \tilde{x}_1 c)^2 + wc^2\right]^{-s+(1-r)/2} \\ & \times k_{1,1}\left(r \mid I_2, \pi k\sqrt{\frac{t(z + \tilde{x}_1 c)^2 + wc^2}{v d_2^2}}\right) dz. \end{aligned}$$

As in the proof of Theorem 3, set $u = \sqrt{t/(wc^2)}\,(z + \tilde{x}_1 c)$ and use formula (1.13) to obtain

$$\begin{aligned} T = {} & \chi(d_2, k) d_2^{-2s+r-1} c^{-2s-r+3/2} \exp\{2\pi i c(d_1\tilde{x}_1 + \tilde{x}_2 k/d_2)\} \\ & \times v^{-s-(1-r)/2} w^{1-s-r/2} t^{-1/2} k_{2,1}\left(s - r/2, r \mid I_3, \pi\sqrt{w}\left(\frac{c}{\sqrt{t}}\left(\frac{kp}{d_2} - d_1\right), \frac{kc}{\sqrt{v}\, d_2}\right)\right). \end{aligned}$$

Set

$$M = \begin{pmatrix} t^{-1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix}\begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix}.$$

Then ${}^tMM = \tilde{U}^{-1}$ and if we set ${}^tm = c(d_1, k/d_2)$, formula (1.12) says that

$$\begin{aligned} T = {} & \chi(d_2, k) d_2^{r-2s-1} c^{2-2s-r} \exp\{2\pi i\, {}^t x A m\} \\ & \times k_{2,1}\left(s - r/2, r \mid \begin{pmatrix} U^{-1}[{}^tA^{-1}] & 0 \\ 0 & w \end{pmatrix}, \pi m\right). \end{aligned}$$

For, again the power functions of $\tilde{U}^{-1}$ will cancel. Q.E.D.

Finally, we consider the case that φ_r in $Z(s,\varphi_r \mid Y)$ is itself an Eisenstein series.

THEOREM 5. *Suppose that* $\varphi_r = \pi^{-r}\Gamma(r)Z(r \mid Y)$ *with* $Z(r \mid Y) =$ *Epstein's zeta function of* (1.2) *and* (2.10) *if*

$$Y = \begin{pmatrix} U & 0 \\ 0 & w \end{pmatrix}\begin{bmatrix} I_2 & x \\ 0 & 1 \end{bmatrix}$$

then the Fourier expansion of $Z(s,\varphi_r \mid Y)$ *as a periodic function of* $x \in \mathbf{R}^2$ *is*

$$\begin{aligned}
&\pi^{-(s-r/2)}\Gamma(s-r/2)\pi^{-(s-(1-r)/2)}\Gamma(s-(1-r)/2)Z(s,\varphi_r \mid Y) \\
&\quad = c(s,r) + c((6-2s-3r)/4, s-r/2) \\
&\qquad + c((3+3r-2s)/4, s-(1-r)/2) \\
&\qquad + \sum_{\substack{k=0 \\ A,c,d_1\neq 0,d_2}} \alpha_0' c^{2-2s-r} d_2^{r-2s} \exp\{2\pi i {}^t x A m\} \\
&\qquad \times k_{2,1}\left(s-r/2, r \,\Big|\, \begin{pmatrix} U[{}^tA^{-1}] & 0 \\ 0 & w \end{pmatrix}, \pi m\right) \\
&\qquad + \sum_{\substack{k=0 \\ A,c,d_1\neq 0,d_2}} \alpha_0 c^{1-2s+r} d_2^{1-r-2s} \exp\{2\pi i {}^t x A m\} \\
&\qquad \times k_{2,1}\left(s-(1-r)/2, 1-r \,\Big|\, \begin{pmatrix} U[{}^tA^{-1}] & 0 \\ 0 & w \end{pmatrix}, \pi m\right) \\
&\qquad + \sum_{\substack{k\neq 0 \\ A,c,d_1,d_2}} \alpha_k c^{2-2s-r} d_2^{r-2s} \exp\{2\pi i {}^t x A m\} \\
&\qquad \times k_{2,1}\left(s-r/2, r \,\Big|\, \begin{pmatrix} U[{}^tA^{-1}] & 0 \\ 0 & w \end{pmatrix}, \pi m\right).
\end{aligned}$$

Here

$$\alpha_0 = \Lambda(s,r)/B(\tfrac{1}{2}, \tfrac{1}{2}-r), \qquad \alpha_0' = \Lambda(s,r)/B(\tfrac{1}{2}, r-\tfrac{1}{2}),$$

$$\alpha_k = \Lambda(s,r)\sigma_{1-2r}(k)/\zeta(2r),$$

$$\Lambda(s,r) = \pi^{-(s-r/2)}\Gamma(s-r/2)\pi^{-(s-(1-r)/2)}\Gamma(s-(1-r)/2),$$

$$c(s,r) = \Lambda(r)\Lambda(s-r/2)\Lambda(s-(1-r)/2)E(r \mid U^0)\,|U|^{-s},$$

$$\Lambda(r) = \pi^{-r}\Gamma(r)\zeta(2r),$$

$$E(r \mid U^0) = \frac{1}{2}\sum_{\gcd(a)=1} U^0[a]^{-r}, \quad \operatorname{Re} r > 1.$$

The 3 *sums in the formula are over*

$$A \in \mathrm{SL}(2,\mathbf{Z})/P(1,1),$$

$P(1,1) =$ *the parabolic subgroup defined by* (1.14), $c > 0$, $d_1 \in \mathbf{Z}$ $(d_1 \neq 0$ *in the* 1*st two sums*$)$, $d_2 > 0$, $d_2 \mid k$, $k \in \mathbf{Z}$ $(k \neq 0$ *in the* 3*rd sum*$)$, *and the vector* $m \in \mathbf{Z}^2$ *is defined to be* ${}^t m = c(d_1, k/d_2)$.

Proof. The proof is the same as that of Theorem 4 except that α_0 and α_0' are not zero. We need to use formulas (2.7) and (2.10). The constant term in the expansion is

$$\Lambda(s,r)\,|U|^{-s}E\big(r\,|\,U^0\big)L\varphi_r(2s) + \alpha_0 k_{2,1}(s-(1-r)/2)\,|\,I_3,0)$$
$$\times \sum_{A,c,d_2} d_2^{-2s+1-r}c^{-2s+1+r}\,|U|^{3(1-r)/4+s/2-3/2}U^{-1}[a_1]^{-s+(1-r)/2}$$
$$+\alpha_0' k_{2,1}(s-r/2,r\,|\,I_3,0)\sum_{A,c,d_2>0} d_2^{r-2s}c^{2-2s-r}\,|U|^{3r/4+s/2-3/2}U^{-1}[a_1]^{-s+r/2}.$$

The computation of Harish-Chandra's c-function (cf. formula (1.3.9) of Terras [**23**]) shows that

$$k_{2,1}(s-r/2,r\,|\,I_3,0) = B\big(\tfrac12, r/2+s-1\big)B\big(\tfrac12, r-\tfrac12\big).$$

And the theory of Hecke operators for GL(2, **Z**) (cf. Terras [**24**]) says that

$$L\varphi_r(2s) = \zeta(2s+r+1)\zeta(2s-r).$$

So the first part of the constant term is indeed $c(s,r)$. The third part of the constant term is

$$\Lambda(s,1-r)B\big(\tfrac12, r/2+s-1\big)B\big(\tfrac12, r-\tfrac12\big)\zeta(2s-r)\zeta\big(2s+\tfrac32-r\big)$$
$$\times|U|^{-(3/2-s/2-3r/4)}E\big(s-r/2\,|\,U^0\big)/B\big(\tfrac12, r-\tfrac12\big)$$
$$= c((6-2s-3r)/4, s-r/2).$$

The second part of the constant term must therefore be

$$c((3-2s+3r)/4, s-(1-r)/2).$$

The rest follows from Theorem 4. Q.E.D.

Let us finish with some remarks on the constant term in Theorem 5. Let $s\in\mathbf{C}^3$, $Y\in P_3$,

$$e_3(s\,|\,Y) = \sum_{A\in \mathrm{SL}(3,\mathbf{Z})/P(1,1,1)} P_s(Y[A])^{-1}, \qquad \operatorname{Re} s_j > 1.$$

Here $P(1,1,1)$ denotes the parabolic subgroup of all upper triangular matrices in SL(3, **Z**). It is easily seen that

$$E(s,\varphi_r\,|\,Y) = e_3(r, s-r/2, 0\,|\,Y),$$

if $\varphi_r(U) = E(r\,|\,U) = Z(r\,|\,U)/\rho(2r)$. So it is natural to consider Selberg's change of variables (cf. Terras [**18**])

$$r = z_2 - z_1 + 1/2, \quad s - r/2 = z_3 - z_2 + 1/2.$$

Then $s = z_3 - (z_1+z_2)/2 + 3/4$. Let

$$r' = s + (r-1)/2 = z_3 - z_1 + 1/2 = r((23)z),$$
$$s' = 3/4 + 3r/4 - s/2 = s((23)z)$$

with $(23)(z_1,z_2,z_3) = (z_1,z_3,z_2)$. Similarly,

$$\tilde r = s - r/2 = ((13)z),$$
$$\tilde s = 3/2 - 3r/4 - s/2 = s((13)z).$$

So the three terms in the 0th Fourier coefficient of $E_{2,1}(s, \varphi_r \mid Y)$ correspond to the permutations (1), (23), (13) of the 2-variables. This is a very general phenomenon described by Langlands in **[12]**.

QUESTIONS REMAINING TO BE ANSWERED. 1. Find similar applications to those mentioned after (1.3); e.g. generalize the Kronecker limit formula.

2. Generalize the formulas to $\mathrm{GL}(n, \mathbf{Z})$. This requires some more work on K-Bessel functions, particularly, more general versions of (1.13) relating $k_{2,2}$ and integrals of products of two $k_{1,1}$'s.

3. Do all automorphic forms for $\mathrm{GL}(3, \mathbf{Z})$ have such Fourier expansions? This probably follows from work of Goodman and Wallach on the uniqueness of such special functions.

4. Can one use such Fourier expansions to develop a classical approach to Hecke theory for $\mathrm{GL}(3, \mathbf{Z})$ parallel to the adelic work of Jacquet, Piatetski-Shapiro, and Shalika in **[9]**?

Note that the following integral diverges:

$$\int_{U \in P_2/\mathrm{SL}(2,\mathbf{Z})} |U|^{-r} \varphi(U^0) k_{2,1}\left(s \,\Big|\, \begin{pmatrix} U & 0 \\ 0 & w \end{pmatrix}, n\right) e^{-\mathrm{Tr}\, U} \frac{dU}{|U|^{3/2}},$$

$$\text{for } r \in \mathbf{C}, \varphi \in \mathcal{Q}(\mathrm{SL}(2, \mathbf{Z}), \chi).$$

Thus it appears that any Mellin transform theory for $\mathrm{GL}(3, \mathbf{Z})$ must involve only a one-dimensional Mellin transform. This may also say that cusp forms for $\mathrm{GL}(3, \mathbf{Z})$ can be as strange as those discovered by Howe and Piatetski-Shapiro for other groups.

BIBLIOGRAPHY

1. W. L. Baily, *Introductory lectures on automorphic forms*, Princeton Univ. Press, Princeton, N. J., 1973.
2. P. T. Bateman and E. Grosswald, *On Epstein's zeta function*, Acta Arith. **9** (1964), 365–373.
3. T. Bengtson, *Matrix K-Bessel functions*, Ph. D. thesis, University of California, San Diego, 1981.
4. S. Chowla and A. Selberg, *On Epstein's zeta functions*, J. Reine Angew. Math. **227** (1967), 86–110.
5. C. W. Curtis, *Representations of finite groups of Lie type*, Bull. Amer. Math. Soc. (N. S.) **1** (1979), 721–757.
6. G. H. Hardy, *Ramanujan*, Chelsea, New York, 1940.
7. D. R. Heath-Brown and S. J. Patterson, *The distribution of Kummer sums at prime arguments*, J. Reine Angew. Math. **310** (1979), 111–130.
8. K. Imai, *Generalization of Hecke's correspondence to Siegel's modular forms*, Amer. J. Math. **102** (1980), 903–936.
9. H. Jacquet, I. I. Piatetski-Shapiro and J. Shalika, *Automorphic forms on GL* (3). I, II, Ann. of Math. (2) **109** (1979), 169–212, 213–258.
10. T. Kubota, *Elementary theory of Eisenstein series*, Wiley, New York, 1973.
11. ______, *Some results concerning reciprocity law and real analytic automorphic functions*, Proc. Sympos. Pure Math., vol. 20, Amer. Math. Soc., Providence, R. I., 1971, pp. 382–395.
12. R. P. Langlands, *Eisenstein series*, Lecture Notes in Math., vol. 541, Springer-Verlag, New York, 1976.
13. H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funcktionalgleichung*, Math. Ann. **121** (1949), 141–183.
14. ______, *Siegel's modular forms and Dirichlet series*, Lecture Notes in Math., vol. 216, Springer-Verlag, New York, 1971.
15. A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47–87.

16. C. L. Siegel, *Einführung in die Theorie der Modulfuncktionen n-ten Grades*, Math. Ann. **116** (1939), 617–637. [Ges. Abh., Vol. II, pp. 97–137.]

17. H. M. Stark, *On the problem of unique factorization in complex quadratic fields*, Proc. Sympos. Pure Math., Vol. 7, Amer. Math. Soc., Providence, R. I., 1969, pp. 41–56.

18. A. Terras, *Fourier analysis on symmetric spaces and applications*, lecture notes.

19. ______, *Applications of special functions for the general linear group to number theory*, Sém. Delange-Pisot-Poitou, 1976/77, no. 23.

20. ______, *Bessel series expansions of the Epstein zeta function and the functional equations*, Trans. Amer. Math. Soc. **183** (1973), 477–486.

21. ______, *Fourier coefficients of Eisenstein series of one complex variable for the special linear group*, Trans. Amer. Math. Soc. **205** (1975), 97–114.

22. ______, *Noneuclidean harmonic analysis*, SIAM Rev. **24** (1982).

23. ______, *Analysis on positive matrices as it might have occurred to Fourier*, Lecture Notes in Math., vol. 899, Springer, Berlin and New York, 1981.

24. ______, *On automorphic forms for the general linear group*, Rocky Mountain J. Math. **12** (1982).

25. W. Weber, *Lehrbuch der Algebra*. III, 3rd ed., Chelsea, New York, 1961.

26. A. Weil, *Elliptic functions according to Eisenstein and Kronecker*, Springer-Verlag, New York, 1976.

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